$$\begin{cases} u_t = a^2\Delta_3 u & 0\le r<r_0 \quad 0\le\varphi<2\pi \quad -\infty<z<+\infty,\ t>0 \\ u|_{t=0}=0 \\ u|_\Gamma = Ae^{-\omega^2 t} \end{cases}$$

Ответ:

$$\boxed{u = Ae^{-\omega^2 t} + \sum_{k=1}^{\infty}\left[\frac{2A}{\mu_0^k J_0'(\mu_0^k)} J\left(\frac{\mu_0^k r}{r_0}\right) e^{-\left(\frac{\mu_0^k a}{r_0}\right)^2 t} + J_0\left(\frac{\mu_0^k r}{r_0}\right)\frac{2A\omega^2}{\mu_0^k J_0'(\mu_0^k)}\left(\frac{e^{-\omega^2 t} - e^{-\left(\frac{\mu_0^k a}{r_0}\right)^2 t}}{\left(\frac{\mu_0^k a}{r_0}\right)^2-\omega^2}\right)\right]}$$

$$u = v + Ae^{-\omega^2 t}$$

$$\begin{cases} v_t = a^2\Delta_3 u + A\omega^2 e^{-\omega^2 t} \\ v|_{t=0} = -A \\ v|_\Gamma = 0 \end{cases}$$

$$v = w + y$$

$$\begin{cases} w_t = a^2\Delta_3 w \\ w|_{t=0} = -A \\ w|_\Gamma = 0 \end{cases}$$

$$w = R(r)\cdot T(t)$$

$$\frac{\Delta_3 R}{R} = \frac{T'}{a^2 T} = -\lambda$$

1) $r^2R'' + rR' + \lambda r^2 R = 0$

$R = J_0\left(\frac{\mu_0^k r}{r_0}\right)$

$J_0(\mu_0^k) = 0$

$\lambda_k = \left(\frac{\mu_0^k}{r_0}\right)^2$

2) $T' + a^2\lambda T = 0$

$T = e^{-\left(\frac{\mu_0^k a}{r_0}\right)^2 t}$

3) $w = \sum_{k=1}^{\infty} A_k J_0\left(\frac{\mu_0^k r}{r_0}\right) e^{-\left(\frac{\mu_0^k a}{r_0}\right)^2 t}$

$w|_{t=0} = \sum_{k=1}^{\infty} A_k J_0\left(\frac{\mu_0^k r}{r_0}\right) = -A$

$$A_k = \int_0^{r_0} r\cdot(-A)\cdot J_0\left(\frac{\mu_0^k r}{r_0}\right)dr\cdot\frac{2}{r_0\left(J_0'(\mu_0^k)\right)^2} =$$

$$= -A\left(\frac{r_0}{\mu_0^k}\right)^2\mu_0 J_1(\mu_0^k) = \frac{2A}{\mu_0 J_0'(\mu_0^{(k)})}$$

$$w = \sum_{k=1}^{\infty}\frac{2A}{\mu_0^k J_0'(\mu_0^{(k)})}\cdot J_0\left(\frac{\mu_0^k r}{r_0}\right)e^{-\left(\frac{\mu_0^k a}{r_0}\right)^2 t}$$

$$\begin{cases} y_t = a^2\Delta_3 y + A\omega^2 e^{-\omega^2 t} \\ y|_{t=0} = 0 \\ y|_\Gamma = 0 \end{cases}$$

Найдем сф. для однор. ур-не, разложим $A\omega^2 e^{-\omega^2 t}$ в ряд по ним

$$\sum_{k=1}^{\infty} J_0\left(\frac{\mu_0^k r}{r_0}\right)T_k' = -\sum_{k=1}^{\infty}\left(\frac{a\mu_0^k}{r_0}\right)^2 J_0\left(\frac{\mu_0^k r}{r_0}\right)T_k + \sum_{k=1}^{\infty}\frac{-2A\omega^2 e^{-\omega^2 t}}{\mu_0^k J_0'(\mu_0^k)}J_0\left(\frac{\mu_0^k r}{r_0}\right)$$

$$T_k' = -\left(\frac{a\mu_0^k}{r_0}\right)^2 T_k - \frac{2A\omega^2 e^{-\omega^2 t}}{\mu_0^k J_0'(\mu_0^k)}, \quad T_k(0) = 0$$

$T_k' = -C_k T_k - D_k e^{-\omega^2 t}$

$T_k = e^{-C_k t}\cdot E_k(t)$

$E_k'(t)e^{-C_k t} = -D_k e^{-\omega^2 t}$

$E_k(t) = \int D_k e^{-\omega^2 t + C_k t}\,dt$

$E_k(t) = \frac{D_k e^{-\omega^2 t + C_k t}}{-\omega^2 + C_k} + F_k$

$T_k = e^{-C_k t}\left(\frac{D_k e^{-\omega^2 t + C_k t}}{C_k - \omega^2} + F_k\right), \quad T_k(0) = 0$

$$T_k = e^{-C_k t}\left(\frac{D_k e^{-\omega^2 t + C_k t}}{C_k - \omega^2} - \frac{D_k}{C_k-\omega^2}\right) =$$

$$= e^{-\left(\frac{a\mu_0^k}{r_0}\right)^2 t}\cdot\left(\frac{2A\omega^2}{\mu_0^k J_0'(\mu_0^k)}\right)\cdot\left(\frac{e^{\left(\frac{a\mu_0^k}{r_0}\right)^2 t - \omega^2 t} - 1}{\left(\frac{a\mu_0^k}{r_0}\right)^2 - \omega^2}\right)$$

$$y = \sum_{k=1}^{\infty} J_0\left(\frac{\mu_0^k r}{r_0}\right)\cdot\left(\frac{2A\omega^2}{\mu_0^k J_0'(\mu_0^k)}\right)\cdot\left(\frac{e^{-\omega^2 t} - e^{-\left(\frac{a\mu_0^k}{r_0}\right)^2 t}}{\left(\frac{a\mu_0^k}{r_0}\right)^2 - \omega^2}\right)$$

$$\begin{cases} u_t = a^2 \Delta_2 u + Axy, & 0<x<l_1,\ 0<y<l_2,\ t>0 \quad A = const \\ u|_{y=0} = u_x|_{x=0} = u|_{x=l_1} = u_y|_{y=l_2} = 0 \\ u(x,y,0) = 0 \end{cases}$$

$u = W(x,y) \cdot T(t)$

Одноф: $u_t = a^2 \Delta_2 u$

$W T' = a^2 \Delta_2 W \cdot T$

$\dfrac{\Delta_2 W}{W} = \dfrac{T'}{a^2 T} = -\lambda$

1) $\Delta_2 W = -\lambda W$

$W = X(x) \cdot Y(y)$

$X'' Y + X \cdot Y'' = -\lambda XY$

$\underbrace{\dfrac{X''}{X}}_{-\mu} + \underbrace{\dfrac{Y''}{Y}}_{-\nu} = -\lambda$

2) $X'' = -\mu X, \quad X'(0) = 0, \quad X(l_1) = 0$

$X = A\cos\sqrt{\mu}x + B\sin\sqrt{\mu}x$

$X'(0) = 0 \Rightarrow B = 0$

$X(l_1) = 0 \Rightarrow \cos\sqrt{\mu} l_1 = 0$

$\sqrt{\mu} l_1 = \frac{\pi}{2} + \pi n$

$\mu_n = \left(\frac{\pi + 2\pi n}{2l_1}\right)^2, n = 0, 1, \ldots$

$X_n = \cos\left(\frac{\pi + 2\pi n}{2l_1} x\right)$

3) $Y'' = -\nu Y, \quad Y(0) = 0, \quad Y'(l_2) = 0$

$Y = A\cos\sqrt{\nu}y + B\sin(\sqrt{\nu}y)$

$Y(0) = 0 \Rightarrow A = 0$

$Y'(l_2) = 0 \Rightarrow \cos\sqrt{\nu}y = 0$

$\sqrt{\nu} l_2 = \frac{\pi}{2} + \pi k$

$\nu_k = \left(\frac{\pi + 2\pi k}{2l_2}\right)^2 \quad k = 0, 1 \ldots$

$Y_k = \sin\left(\frac{\pi + 2\pi k}{2l_2} y\right)$

$\lambda_{nk} = \left(\frac{\pi + 2\pi n}{2l_1}\right)^2 + \left(\frac{\pi + 2\pi k}{2l_2}\right)^2 \quad n, k = 0, 1, \ldots$

$u = \sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{\infty} X_n \cdot Y_k \cdot T_{nk}$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} X_n \cdot Y_k \cdot T'_{nk} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (-\lambda_{nk} a^2) X_n Y_k T_{nk} + Axy$$

Разложим Axy в ряд по X_n, Y_k

$$F_{nk} = \frac{2}{l_1}\cdot\frac{2}{l_2}\cdot\int_0^{l_1}\int_0^{l_2} Axy\cdot\cos\left(\frac{\pi+2\pi n}{2l_1}x\right)\cdot\sin\left(\frac{\pi+2\pi k}{2l_2}y\right)dx\,dy$$

$$\int_0^{l_1} x\cos\left(\frac{\pi+2\pi n}{2l_1}x\right)dx = x\frac{\sin\left(\frac{\pi+2\pi n}{2l_1}x\right)}{\left(\frac{\pi+2\pi n}{2l_1}\right)}\Bigg|_0^{l_1} - \int_0^{l_1}\frac{\sin\left(\frac{\pi+2\pi n}{2l_1}x\right)}{\frac{\pi+2\pi n}{2l_1}}dx =$$

$$= \frac{l_1\sin\left(\frac{\pi+2\pi n}{2l_1}l_1\right)}{\frac{\pi+2\pi n}{2l_1}} + \frac{\cos\left(\frac{\pi+2\pi n}{2l_1}x\right)}{\left(\frac{\pi+2\pi n}{2l_1}\right)^2}\Bigg|_0^{l_1} = \frac{2l_1^2 (-1)^n}{\pi+2\pi n} - \frac{4l_1^2}{(\pi+2\pi n)^2}$$

$$\int_0^{l_2} y\sin\left(\frac{\pi+2\pi k}{2l_2}y\right)dy = -y\frac{\cos\left(\frac{\pi+2\pi k}{2l_2}y\right)}{\frac{\pi+2\pi k}{2l_2}}\Bigg|_0^{l_2} + \int_0^{l_2}\frac{\cos\left(\frac{\pi+2\pi k}{2l_2}y\right)}{\frac{\pi+2\pi k}{2l_2}}dy =$$

$$= 0 + \frac{\sin\left(\frac{\pi+2\pi k}{2l_2}y\right)}{\left(\frac{\pi+2\pi k}{2l_2}\right)^2}\Bigg|_0^{l_2} = \frac{\sin\left(\frac{\pi}{2}+\pi k\right)}{\left(\frac{\pi+2\pi k}{2l_2}\right)^2} = \frac{(-1)^k 4l_2^2}{(\pi+2\pi k)^2}$$

$$\Rightarrow \boxed{F_{nk} = 4A\left(\frac{2l_1(-1)^n}{\pi+2\pi n} - \frac{4l_1}{(\pi+2\pi n)^2}\right)\left(\frac{(-1)^k\cdot 4l_2}{(\pi+2\pi k)^2}\right)}$$

Тогда $T'_{nk} = -\lambda_{nk}a^2 T_{nk} + F_{nk}$, где F_{nk} не зависит от t

$T_{nk}(0) = 0$

$$T_{nk} = C_{nk}e^{-\lambda_{nk}a^2 t} + \frac{F_{nk}}{\lambda_{nk}a^2}$$

$$T_{nk}(0) = C_{nk} + \frac{F_{nk}}{\lambda_{nk}a^2} = 0 \Rightarrow C_{nk} = -\frac{F_{nk}}{\lambda_{nk}a^2}$$

Ответ: $$u = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\cos\left(\frac{\pi+2\pi n}{2l_1}x\right)\sin\left(\frac{\pi+2\pi k}{2l_2}y\right)\cdot\frac{F_{nk}}{\lambda_{nk}a^2}\left(-e^{-\lambda_{nk}a^2 t}+1\right),$$

где $\lambda_{nk} = \left(\frac{\pi+2\pi n}{2l_1}\right)^2 + \left(\frac{\pi+2\pi k}{2l_2}\right)^2$, а F_{nk} найдено выше

№ 3

$$\begin{cases} u_t = a^2 \Delta_3 u \\ u|_{t=0} = 0 \\ u|_{\Gamma} = u_0 \end{cases} \qquad 0 \le r < r_0, \quad 0 \le \varphi < 2\pi \quad -\infty < z < +\infty, \quad t > 0$$

$$u = u_0 + v$$

$$\begin{cases} v_t = a^2 \Delta_3 v \\ v|_{t=0} = -u_0 \\ v|_{\Gamma} = 0 \end{cases}$$

– уже решали такую задачу в № 1 для w

Ответ: $$\boxed{u = u_0 + \sum_{k=1}^{\infty} \frac{2A}{\mu_0^k \, J_0'(\mu_0^k)} \cdot J_0\left(\frac{\mu_0^k r}{r_0}\right) e^{-\left(\frac{\mu_0^k a}{r_0}\right)^2 t}}$$

$J_0(\mu_0^k) = 0.$